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A SIMPLE ANALYTIC METHOD FOR COMPUTING INSTRUMENT POINTING JITTER

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ABSTRACT

Pointing jitter refers to camera motion occurring during an image exposure of specified duration. Jitter acts to smear optical images and degrade the quality of pictures taken by scientific instruments. As such it is an important measure of a controller's effectiveness, and an important quantity to characterize mathematically. Unfortunately, the numerical calculation of pointing jitter involves the integral of a rational polynomial multiplied by a transcendental weighting function. This form has restricted its evaluation to purely numerical methods, and complicated its use in practice. The present paper overcomes this limitation by introducing a state-space method for evaluating the integral. The state-space expression is applicable to a wide range of pointing processes used in practice (i.e., stationary processes with arbitrary rational spectrum), and completely avoids numerical integration.

1 INTRODUCTION

This paper is concerned with statistical analysis of instrument pointing control performance. A recent definition of RMS pointing jitter put forth in Sirlin and San Martin [10] (see also Appendix A of Lucke, Sirlin and San Martin [8]), will be reviewed which captures the dependence of image smear on the duration of a finite-time exposure window. This dependence is critical to correctly capturing the "fast film" benefit, which says that image smear in a camera disturbed by motion having a low-pass power spectrum, can be made arbitrarily small by taking exposures of sufficiently short duration. The RMS pointing jitter expression discussed here nicely captures this effect under very general conditions. (In an interesting contrast to imaging instruments, a recent pointing criterion for spectroscopic instruments [2] shows that performance actually improves as the exposures become *long*).

Presently, evaluation of the RMS pointing jitter is based on a frequency domain integral involving a rational polynomial multiplied by a transcendental weighting function [10][8]. Unfortunately, this form has restricted its evaluation to purely numerical methods, and complicated its use in practice. The present paper will overcome this limitation by introducing a state-space method for evaluating the integral. The state-space expression is applicable to a wide range of pointing processes used in practice (i.e., stationary processes with arbitrary rational spectrum), and completely avoids numerical integration.

2 INSTRUMENT POINTING JITTER

2.1 Background

Physically, a pointing process is defined by the motion that a camera or instrument boresight undergoes as a function of time. While this motion most generally forms a two-dimensional process (both up-down, and left-right), we will intentionally restrict ourselves to the one-dimensional case arising from the projection of this motion onto a single axis.

It will be assumed that the pointing process defined above can be suitably approximated by a second-order stationary random process (cf., Papoulis [7]). This permits all of the pointing control definitions to be made in precise mathematical terms.

The second-order stationarity assumption is relatively mild in the sense that it requires only that the process is stationary in its first and second moments (as opposed to strict-sense stationarity which requires stationarity in all moments) and does not impose any specific form on the shape of the underlying probability distributions.

2.2 Pointing Definitions

The per-axis pointing control definitions of interest are depicted graphically in Figure 2.1. This diagram will be discussed in detail in this section.

Let $n(t)$ be a zero-mean second-order stationary random process with power spectral density $S_n(\omega)$ such that,

$$E[n(t)] = 0 \tag{2.1}$$

$$R_n(\ell) = E[n(t)n(t+\ell)] \tag{2.2}$$

$$S_n(\omega) = F\{R_n(\ell)\} \tag{2.3}$$

$$\sigma_a^2 = R_n(0) \tag{2.4}$$

Here, the Fourier Transform of a signal $x(t)$ is denoted as $X(\omega) = F\{x(t)\}$ and is defined as,

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{2.5}$$

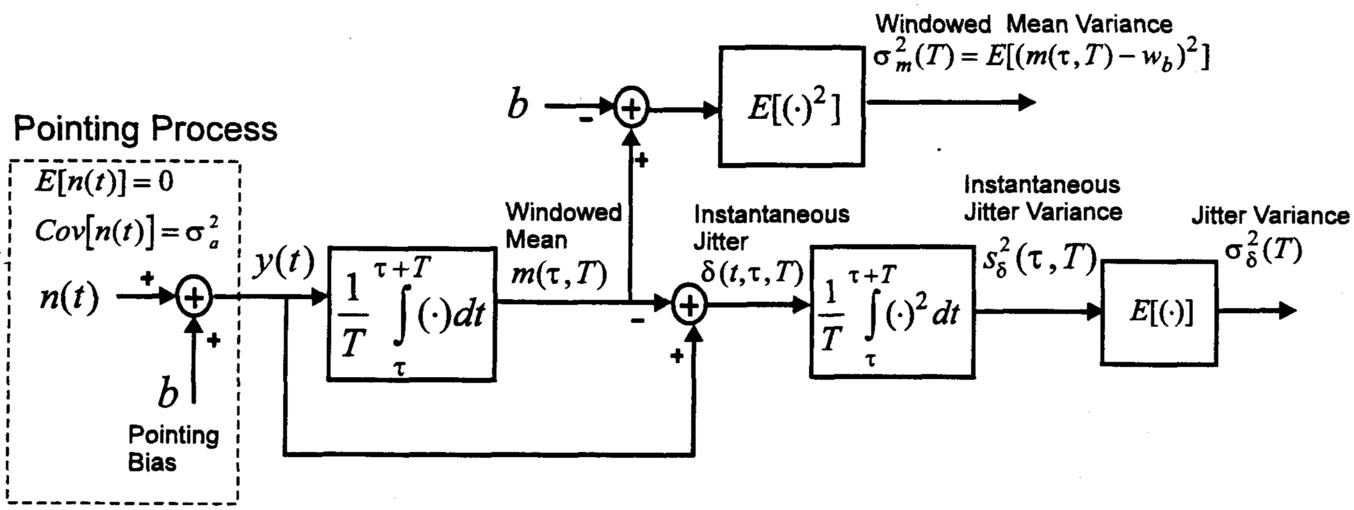

Figure 2.1: Pointing control definition diagram

Let b be a constant bias. Then the pointing process $y(t)$ is defined as,

$$y(t) = n(t) + b \tag{2.6}$$

DEFINITION 2.1 *The random process $y(t)$ of the form (2.6) defines a* **pointing process**. ■

DEFINITION 2.2 σ_a^2 *is the* **pointing process variance**. ■

DEFINITION 2.3 b *is the* **pointing bias**. ■

DEFINITION 2.4 *The* **windowed-mean** *over a window of duration T, starting at time τ is defined as,*

$$m(\tau, T) = \frac{1}{T}\int_\tau^{\tau+T} x(t)dt \tag{2.7}$$

■

The windowed-mean $m(\tau, T)$ can be thought of as ar estimate of the process mean b which improves as the window duration T becomes large. This type of estimate is used in many instrument calibration functions (e.g., frame misalignments, etc.) which are designed to estimate b as accurately as possible from measurements on a finite observation interval.

The windowed-mean $m(\tau, T)$ is a random quantity which depends on the starting time τ and the particular realization of the random process. Its variance based on an **ensemble average** is defined next.

DEFINITION 2.5 *The* **windowed-mean variance** *for windows of size T, is defined as,*

$$\sigma_m^2(T) = E\left[(m(\tau, T) - b)^2\right] \tag{2.8}$$

■

Here, the notational dependence of σ_m^2 on τ has been dropped since the ensemble average in (2.8) is taken with respect to y which is a stationary process.

When an image is taken over an exposure of duration T, it is of interest to know how much smearing occurred due to the instrument boresight "jittering around" during the exposure. This jitter effect is captured by the following definition.

DEFINITION 2.6 *The* **instantaneous jitter** *over a window starting at time τ and ending at time $\tau + T$ is defined as,*

$$\delta(t, \tau, T) = y(t) - m(\tau, T) \tag{2.9}$$

■

The definition of instantaneous jitter is shown pictorially in Figure 2.2. It is important to note that the jitter is defined as *the instantaneous deviation of the pointing process from the windowed-mean $m(\tau, T)$ rather than from the process mean b.* This unusual definition turns out to be critical to correctly capturing the effect of camera motion on image smearing. Intuitively, the reason is that an image taken on a finite interval $t = [\tau, \tau+T]$ will be collecting photons according to $y(t)$ only on this interval, and hence be centered at the windowed-mean rather than the process mean. Accordingly, deviations from the windowed-mean cause smear, rather than deviations from the process mean.

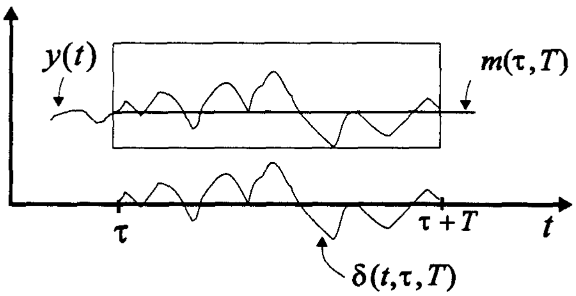

Figure 2.2: Definition of instantaneous jitter $\delta(t, \tau, T)$

The earliest rigorous analytical characterization of the instantaneous jitter as defined above, appears in Sirlin and San Martin [10], and subsequently by Lucke, Sirlin and San Martin [8].

As shown in Figure 2.2 the three time variables t, τ, T are essential to properly describe δ. Specifically, t is the plotting variable, the window starts at $t = \tau$, and ends at $t = \tau + T$.

The overall effect of the instantaneous jitter on image smearing is captured by its variance in time,

DEFINITION 2.7 *The* **instantaneous jitter variance** *is defined as,*

$$s_\delta^2(\tau, T) \triangleq \frac{1}{T} \int_\tau^{\tau+T} \delta^2(t, \tau, T) dt \tag{2.10}$$

■

The term "instantaneous" is used here since $s_\delta^2(\tau, T)$ is a random variable which depends on the starting time τ, and the particular realization of the pointing process. By taking an *ensemble average* of the instantaneous jitter variance (2.10) one obtains the jitter variance defined next.

DEFINITION 2.8 *The* **jitter variance** *is defined as,*

$$\sigma_\delta^2(T) \triangleq E\left[\frac{1}{T}\int_\tau^{\tau+T} \delta^2(t,\tau,T)dt\right] \tag{2.11}$$

■

Here, the notational dependence of σ_δ^2 on τ has been dropped since the ensemble average in (2.11) is taken with respect to y which is a stationary process. It is emphasized that unlike $s_\delta^2(\tau, T)$, the quantity, $\sigma_\delta^2(T)$ is not a random quantity. Rather it has been averaged over the ensemble of possible realizations and is a deterministic function of T.

It is convenient to define the RMS jitter which results from simply taking the square root of (2.11), i.e.,

DEFINITION 2.9 *The* **RMS jitter** *in a window of duration T is defined as,*

$$\sigma_\delta(T) = \sqrt{E\left[\frac{1}{T}\int_\tau^{\tau+T} \delta^2(t,\tau,T)dt\right]} \tag{2.12}$$

■

The RMS jitter is important because it is a statistic of the pointing process which completely characterizes the control performance as it affects most types of imaging instruments. As such, it has been adopted in many recent JPL/NASA space missions for defining pointing requirements (cf., , Cassini [4], the SIRTF telescope [6], the Space Interferometry mission [7], Europa orbiter [3]). For example, the SIRTF mission (to replace Hubble in NASA's Great Space Observatory series) has its level 1 pointing requirements defined as $\sigma_\delta(T = 200\ secs) = .3\ arcsec$ and $\sigma_\delta(T = 500\ secs) = .6\ arcsec$ [6].

2.3 Frequency Domain Integrals

Several important frequency domain integrals have been developed in Sirlin and San Martin [10] (cf., also Appendix A of [8]) for the various jitter expressions defined above. These are summarized below,

$$\sigma_a^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)d\omega \tag{2.13}$$

$$\sigma_m^2(T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)W_T(\omega)d\omega \tag{2.14}$$

$$\sigma_\delta^2(T) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)(1 - W_T(\omega))d\omega \tag{2.15}$$

where the weighting function is given by,

$$W_T(\omega) = \left[\frac{\sin(\omega T/2)}{\omega T/2}\right]^2 \tag{2.16}$$

The weighting function $W_T(\omega)$ acts as a low-pass filter in the windowed-mean variance integral (2.14), and its complement acts like a high-pass filter in the jitter variance integral (2.15). This agrees with one's intuitive notion that the jitter which affects image smear is high-frequency in nature. Let $S_n(\omega)$ be a lowpass process. Since $S_n(\omega)(1 - W_T(\omega)) \to 0$ as $\lim T \to \infty$, one can infer the "fast film" benefit from (2.15), i.e.,

$$\lim_{T\to 0} \sigma_\delta^2(T) = 0 \tag{2.17}$$

This says that the instrument pointing jitter can be made arbitrarily small by taking exposures of sufficiently short duration T.

2.4 Conservation of Variance

By inspection, it is seen that the frequency integral (2.13) can be written as the sum of the integrals (2.14) and (2.15). This gives the following conservation-of-variance formula,

$$\sigma_a^2 = \sigma_m^2(T) + \sigma_\delta^2(T) \tag{2.18}$$

This formula says that the process variance σ_a^2 is a conserved quantity. Specifically, in any window of duration T, the pointing process variance σ_a^2 divides itself (generally unequally) between the windowed-mean variance $\sigma_m^2(T)$ and the jitter variance $\sigma_\delta^2(T)$. Furthermore, the division is such that most of the process variability goes into the windowed-mean for short time exposures, and into the jitter variance for long exposures, i.e.,

$$\sigma_a^2 = \lim_{T\to 0} \sigma_m^2(T) \tag{2.19}$$

$$\sigma_a^2 = \lim_{T\to \infty} \sigma_\delta^2(T) \tag{2.20}$$

Because of the relation (2.20), the process variance σ_a^2 is sometimes referred to as the **long-term jitter**, or **steady-state jitter**.

3 MAIN RESULT

The main result is presented which introduces a state-space method for evaluating the integral (2.14) for the windowed-mean variance $\sigma_m^2(T)$, and the integral (2.15) for the jitter variance $\sigma_\delta^2(T)$, without requiring numerical integration. For this presentation, the pointing bias in (2.6) is assumed to be zero (i.e., $b = 0$) without loss of generality.

THEOREM 3.1 *Let the stationary process $y(t)$ be generated by the following state-space model driven by white noise,*

$$\dot{x} = Ax + w \tag{3.1}$$

$$y = Cx \tag{3.2}$$

Here, $A \in \mathcal{R}^{n\times n}$ is an asymptotically stable matrix (i.e., all eigenvalues have strictly negative real parts), $C \in \mathcal{R}^{1\times n}$ is the output matrix, and $w \in \mathcal{R}^{n\times 1}$ is a continuous-time zero-mean white noise having statistics,

$$E(w(t)] = 0 \tag{3.3}$$

$$E[w(t)w(t+\tau)^T] = \delta(\tau)Q \tag{3.4}$$

Here the covariance $Q \in \mathcal{R}^{n\times n}$ can be either a positive definite or positive semi-definite symmetric matrix.

Then assuming the system (3.1)(3.2) has reached steady-state, the windowed-mean variance $\sigma_m^2(T)$, and the jitter variance $\sigma_\delta^2(T)$ of $y(t)$ on the interval $t \in [\tau, \tau+T]$ can be calculated as,

$$\sigma_m^2(T) = \frac{2}{T^2} C\mathcal{H}_T P_\infty C^T \tag{3.5}$$

$$\sigma_\delta^2(T) = \sigma_a^2 - \sigma_m^2(T) \tag{3.6}$$

where,

$$\sigma_a^2 = CP_\infty C^T \tag{3.7}$$

$$\mathcal{H}_T = \int_0^T \int_0^s e^{A r} dr ds \tag{3.8}$$

and P_∞ is found by solving the Lyapunov equation,

$$0 = AP_\infty + P_\infty A^T + Q \tag{3.9}$$

Furthermore, the quantity $\mathcal{H}_T$ can be evaluated using any of the following expressions without requiring numerical integration,

Method 1: Matrix Exponential

$$\mathcal{H}_T = A^{-2}\left(e^{AT} - I - AT\right) \tag{3.10}$$

Method 2: Inverse Laplace Transform

$$\mathcal{H}_T = \mathcal{L}^{-1}\left\{\frac{1}{s^2}(sI - AT)^{-1}\right\} \tag{3.11}$$

Method 3: Augmented Matrix Exponential

H_T is the upper right $n \times n$ submatrix of the matrix exponential,

$$e^{XT} = \begin{bmatrix} F_1 & G_1 & \mathcal{H}_T \\ 0 & F_2 & G_1 \\ 0 & 0 & F_3 \end{bmatrix} \tag{3.12}$$

where $X \in \mathcal{R}^{3n\times 3n}$ is defined as (denoting $\mathcal{I}$ as the $n \times n$ identity matrix, and $\mathcal{O}$ as an $n \times n$ zero matrix),

$$X = \begin{bmatrix} \mathcal{O} & \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & A \end{bmatrix} \tag{3.13}$$

■

3.1 Discussion

The main usefulness of the state-space formulation in Theorem 3.1 is that it replaces the weighted frequency integrals (2.14)(2.15) in the expressions for windowed-mean variance and jitter variance, with the unweighted time integral of a matrix exponential, i.e., $\mathcal{H}_T$ in (3.8). Aside from eliminating the transcendental weighting function W_T in (2.16) from the problem, the state-space formulation allows one to take advantage of special results available for integrating expressions involving the matrix exponential. Specifically, Theorem 3.1 provides three methods for evaluating $\mathcal{H}_T$ without numerical integration.

REMARK 3.1 A Gaussian assumption has not been required on the noise w in (3.1)(3.2). Accordingly, the results of Theorem 3.1 hold for white noise with arbitrary probability distributions. ■

REMARK 3.2 For some state-space systems of the form (3.1)(3.2) the matrix exponential expression (3.10) or inverse Laplace transform expression (3.11) can be computed in closed-form. For these systems the jitter formula can be computed in closed-form. It will be seen in the next section that several useful results can be obtained in this fashion. ■

REMARK 3.3 Often it is of interest to evaluate jitter for various sized windows T. A useful simplification occurs if the desired T_k lie on a uniformly spaced time grid $T_k = kT_0$ since each matrix exponential in Method 1 can be calculated simply as $e^{AT_k} = (e^{AT_0})^k$. ■

4 APPLICATIONS

The state-space formulas developed in Theorem 3.1 are applied to computing the jitter first for a first-order Gauss-Markov process, and second, to a typical spacecraft pointing process.

4.1 First-Order Pointing Process

This first-order process is very simple, but can be used to approximate the behaviour of many physical processes and phenomena [5]. Its state-space model is given as,

$$\dot{x} = -ax + bw \tag{4.1}$$

$$y = x \tag{4.2}$$

$$E[w(t)w(t+\tau)] = q \cdot \delta(\tau) \tag{4.3}$$

The power spectrum is computed as,

$$S_n(\omega) = q \cdot |F(j\omega)|^2 \tag{4.4}$$

where the coloring filter is defined as,

$$F(j\omega) = \frac{b}{j\omega + a} \tag{4.5}$$

Applying the results of Theorem 3.1 with the choices, one has $A = -a, C = 1, Q = q$, gives the following results.

$$\sigma_a^2 \triangleq \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)d\omega = \frac{q \cdot b^2}{2a} \tag{4.6}$$

$$\sigma_m^2(T) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)W_T(\omega)d\omega = \sigma_a^2 \cdot \frac{2\left(e^{-aT} + aT - 1\right)}{a^2T^2} \tag{4.7}$$

$$\sigma_\delta^2(T) \;=\; \sigma_a^2 - \sigma_m^2(T) = \sigma_a^2 \cdot \left[1 - \frac{2\left(e^{-aT} + aT - 1\right)}{a^2T^2}\right] \tag{4.8}$$

The results are plotted in Figure 4.1 and tabulated in Table 4.1 for visualization. To aid the interpretation, it is pointed out that the autocorrelation of the process is given by,

$$R(\tau) = E[x(t)x(t+\tau)] = \sigma_a^2 \cdot e^{-a\tau} \tag{4.9}$$

i.e., the process is strongly correlated over its *correlation length* of time $\tau_c = 1/a$.

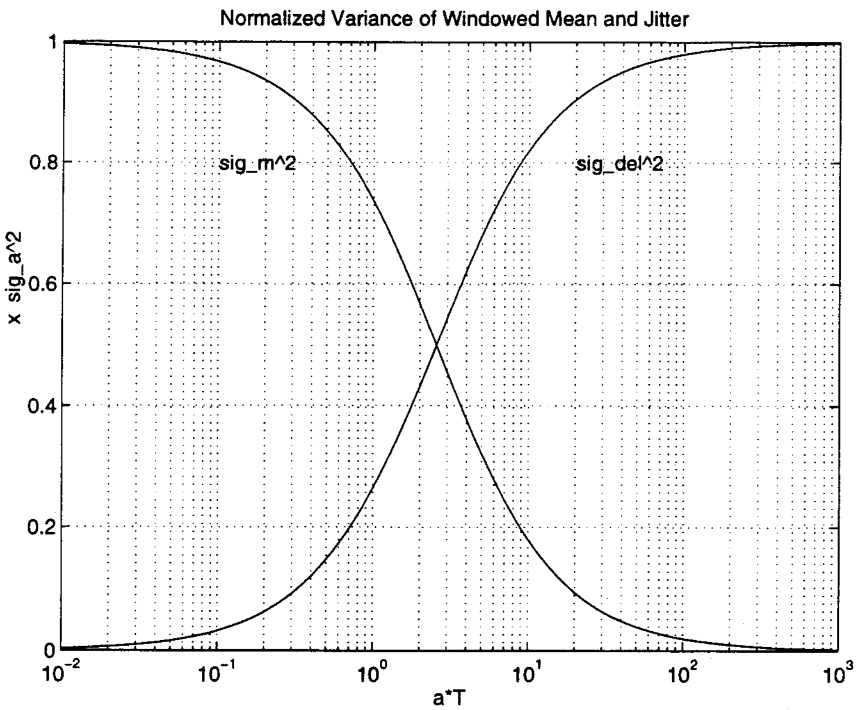

Figure 4.1: Normalized Variance of Windowed Mean $\sigma_m^2(T)/\sigma_a^2$ and Jitter $\sigma_\delta^2(T)/\sigma_a^2$

First-Order Pointing Process Variances

Window Time	Jitter Variance	Mean Variance
T	$\sigma_\delta^2(T)$	$\sigma_m^2(T)$
$0.3247/a$	$.1 * \sigma_a^2$	$.9 * \sigma_a^2$
$0.7101/a$	$.2 * \sigma_a^2$	$.8 * \sigma_a^2$
$1.1795/a$	$.3 * \sigma_a^2$	$.7 * \sigma_a^2$
$1.7720/a$	$.4 * \sigma_a^2$	$.6 * \sigma_a^2$
$2.5569/a$	$.5 * \sigma_a^2$	$.5 * \sigma_a^2$
$3.6734/a$	$.6 * \sigma_a^2$	$.4 * \sigma_a^2$
$5.4483/a$	$.7 * \sigma_a^2$	$.3 * \sigma_a^2$
$8.8732/a$	$.8 * \sigma_a^2$	$.2 * \sigma_a^2$
$18.9443/a$	$.9 * \sigma_a^2$	$.1 * \sigma_a^2$

Table 4.1: Jitter variance and windowed-mean variance for first-order pointing process

It is seen from Table 4.1 that the jitter variance exactly equals the mean variance when the horizon length T equals 2.5569 correlation lengths, i.e., $T = 2.5569 \cdot \tau_c$. For windows T which are short compared to $2.5569 \cdot \tau_c$ the windowed-mean becomes the dominant source of variability. For example, it increases to 90% of the total variance σ_a^2 when T is approximately a third of a correlation length. In contrast, for windows which are long compared to $2.5569 \cdot \tau_c$ the jitter dominates, increasing to a value of 90% at approximately 19 correlation lengths.

4.2 Typical Spacecraft Pointing Process

In this section, an analytic expression will be derived for the jitter variance associated with a three-axis controlled spacecraft using noisy gyro and star tracker measurements. This analysis characterizes control errors due to noisy sensors only. Additional errors due to process noise, and/or environmental torques are assumed to negligibly small, or to be budgeted separately elsewhere. It is also assumed that the spacecraft is flying an attitude estimator comprised of three decoupled single-axis observers. Many spacecraft attitude estimators are of this form or can be reasonably approximated as such.

The single-axis estimation error associated with using a Luenberger observer driven by a star tracker angle measurement and a gyro-based rate measurement, propagates according to the following state-space model [1],

$$\dot{e} = A_o\, e + w_o \tag{4.10}$$

$$e_o = C_o e \tag{4.11}$$

$$A_o = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}; \quad C_o \triangleq \begin{bmatrix} 1 & 0 \end{bmatrix}; \quad K \triangleq \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{4.12}$$

$$Q_o = Q + r \cdot KK^T; \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \tag{4.13}$$

Here, r denotes the star tracker noise covariance. The gyro noise covariances q_1 and q_2 are denoted as the angle random walk (ARW) and bias instability, respectively, and the state is given as $x = [\theta, b]^T$ where θ is the angular position (to be estimated), and b represents the gyro bias.

The characteristic polynomial of the observer with state-space matrix A_o in (4.12) can be calculated as,

$$\det\left(sI - A_o\right) = s^2 + k_1 s + k_2 \tag{4.14}$$

The observer poles are calculated as roots of (4.14), to give,

$$\beta_1 = \frac{-k_1 + \sqrt{k_1^2 - 4k_2}}{2} \tag{4.15}$$

$$\beta_2 = \frac{-k_1 - \sqrt{k_1^2 - 4k_2}}{2} \tag{4.16}$$

Applying the results of Theorem 3.1 with the matrices A_o, C_o, Q_o above, gives the following results,

$$\sigma_a^2 = p_{11} \tag{4.17}$$

$$\sigma_m^2(T) = \frac{2}{T^2}\left(\frac{p_{11}\beta_1 + p_{12}}{\beta_1^2(\beta_1 - \beta_2)} \cdot e^{\beta_1 T} - \frac{p_{11}\beta_2 + p_{12}}{\beta_2^2(\beta_1 - \beta_2)} \cdot e^{\beta_2 T}\right. \tag{4.18}$$

$$\left. + \frac{1}{\beta_1\beta_2} \cdot T + \frac{p_{11}}{\beta_1\beta_2} + \frac{p_{12}(\beta_1 + \beta_2)}{\beta_1^2\beta_2^2}\right) \tag{4.19}$$

$$\sigma_\delta^2(T) = \sigma_a^2 - \sigma_m^2(T) \tag{4.20}$$

where the steady-state covariances are given as,

$$P_\infty \triangleq E[ee^T] = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \tag{4.21}$$

$$p_{11} = \frac{r(k_2^2 + k_1^2 k_2) + q_1 k_2 + q_2}{2k_1 k_2} \tag{4.22}$$

$$p_{12} = \frac{r k_2^2 + q_2}{2k_2} \tag{4.23}$$

$$p_{22} = \frac{r k_2^3 + q_1 k_2^2 + q_2(k_2 + k_1^2)}{2k_1 k_2} \tag{4.24}$$

The above formulas are potentially very useful for analyzing spacecraft pointing system performance in support of a broad range of imaging type instruments.

5 CONCLUSIONS

The RMS pointing jitter criterion has been reviewed as an important statistic of any stationary random pointing process which completely characterizes the control performance as it affects most types of imaging instruments. As such it has been adopted by several recent JPL/NASA missions for specifying basic mission pointing requirements. The main result of this paper is Theorem 3.1 which gives a state-space method for evaluating the instrument pointing jitter without numerical integration. It is hoped that this result will simplify the application of the RMS jitter criterion in practice, and aid its use and adoption.

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A Appendix A: Proof of Theorem 3.1

Assume that at time $t = \tau$ the process x has already reached steady-state, so that its covariance is calculated by solving the Lyapunov equation,

$$AP_\infty + P_\infty A^T + Q = 0 \tag{A.1}$$

where,

$$P_\infty = \lim_{t \to \infty} E[x(t)x(t)^T] \tag{A.2}$$

Note that a solution for P_∞ always exists since the matrix A is asymptotically stable [5]. The formula (3.7) for σ_a^2 (i.e., the steady-state covariance of y) follows directly from (A.1)(A.2) and the output equation (3.2).

Assume that process y is augmented by a single integrator at its output, to give,

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{w} \tag{A.3}$$

$$\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}; \quad \tilde{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \tag{A.4}$$

$$\tilde{w} = \begin{bmatrix} w \\ 0 \end{bmatrix}; \quad \tilde{Q} \triangleq Cov[\tilde{w}] = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \tag{A.5}$$

It will be assumed that the integrator is initialized to zero at time $t = \tau$ so that the state $z(\tau + T)$ is the time integral of y from $t = \tau$ to $t = \tau + T$, i.e.,

$$z(\tau + T) = \int_\tau^{\tau+T} y(r)dr \tag{A.6}$$

It is emphasized that because the model (3.1)(3.2) has reached steady-state by assumption, the signal $y(t)$ being integrated is a realization of a stationary random process.

This construction of z simplifies the calculation of the statistic $\sigma_m^2(T)$ through the relations,

$$m(\tau, T) = \frac{z(\tau + T)}{T} \tag{A.7}$$

$$\sigma_m^2(T) \triangleq E[m(\tau, T)^2] = \frac{P_{zz}(\tau + T)}{T^2} \tag{A.8}$$

where,

$$P_{zz}(\tau + T) \triangleq E[z(\tau + T)^2] \tag{A.9}$$

i.e., it is only left to characterize the covariance P_{zz} of the state z. To this end, the variance of the augmented system (A.3) propagates from time $t = \tau$ to time $t = \tau + T$ according to the following Lyapunov equation,

$$\dot{\tilde{P}}(t) = \tilde{A}\tilde{P}(t) + \tilde{P}(t)\tilde{A} + \tilde{Q} \tag{A.10}$$

where,

$$\tilde{P}(t) \triangleq E[\tilde{x}(t)\tilde{x}(t)^T] = \begin{bmatrix} P_{xx}(t) & P_{xz}(t) \\ P_{xz}(t)^T & P_{zz}(t) \end{bmatrix} \tag{A.11}$$

$$\tilde{P}(\tau) = \begin{bmatrix} P_\infty & 0 \\ 0 & 0 \end{bmatrix} \tag{A.12}$$

The initial condition $\tilde{P}(\tau)$ arises from the fact that the subsystem (3.1)(3.2) is already in steady-state at time $t = \tau$ with covariance P_∞, and the assumption that the integrator is initialized as $z(\tau) = 0$ with probability one, so that $P_{zz}(\tau) = 0$ and $P_{xz}(\tau) = 0$.

The differential equation (A.10) can be written equivalently in terms of the partitioned quantities as follows,

$$\dot{P}_{xx}(t) = AP_{xx}(t) + P_{xx}(t)A^T + Q \tag{A.13}$$

$$\dot{P}_{xz}(t) = AP_{xz}(t) + P_{xx}(t)C^T \tag{A.14}$$

$$\dot{P}_{zz}(t) = 2CP_{xz}(t) \tag{A.15}$$

Using (A.1), one can trivially solve (A.13) with initial condition $P_{xx}(\tau) = P_\infty$ to give,

$$P_{xx}(t) = P_\infty \quad \text{for all } t \in [\tau, \tau + T] \tag{A.16}$$

i.e., the original system (3.1)(3.2) (which starts in steady-state), remains in steady-state throughout the entire interval. Substituting (A.16) into (A.14) gives,

$$\dot{P}_{xz} = AP_{xz} + P_\infty C^T \tag{A.17}$$

Equation (A.17) can be recognized as a system of linear differential equations with constant input, and having a zero initial condition $P_{xz}(\tau) = 0$. As such, its solution at any time t can be expressed in terms of the matrix exponential,

$$P_{xz}(t) = \left[\int_\tau^t e^{A(t-\ell)} d\ell\right] P_\infty C^T = \left[\int_0^{t-\tau} e^{Ar} dr\right] P_\infty C^T \tag{A.18}$$

where use has been made of the change of variable $r = t - \ell$ to simplify the integral. Substituting (A.18) into (A.15) and integrating with respect to time gives,

$$P_{zz}(\tau + T) = 2C\left[\int_\tau^{\tau+T} \int_0^{t'-\tau} e^{Ar} dr dt'\right] P_\infty C^T = 2C\left[\int_0^T \int_0^s e^{Ar} dr ds\right] P_\infty C^T \tag{A.19}$$

where use has been made of the change of variable $s = t' - \tau$ to simplify the integral. Substituting (A.19) into (A.8) gives,

$$\sigma_m^2(T) = \frac{2}{T^2} C\left[\int_0^T \int_0^s e^{Ar} dr ds\right] P_\infty C^T \tag{A.20}$$

which gives the desired expressions (3.5)(3.8).

The expression (3.6) is simply a rearrangement of the conservation of variance formula (2.18).

Method 3 for calculating $\mathcal{H}_T$ from (3.12)(3.13) follows as a special case of Theorem 1 in Van Loan [11]. By expanding e^{XT} in (3.12) into a power series one can extract the series for H_T as the upper right $n \times n$ submatrix to give,

$$H_T = \frac{T^2}{2!}I + \frac{T^3}{3!}A + \frac{T^4}{4!}A^2 + ... \tag{A.21}$$

Multiplying both sides of (A.21) on the left by A^2 and adding $I + AT$ to both sides gives,

$$A^2 H_T + I + AT = I + AT + \frac{T^2}{2!}A^2 + \frac{T^3}{3!}A^3 + \frac{T^4}{4!}A^4 + ... = e^{AT} \tag{A.22}$$

where this infinite series has been recognized as a power series representation of e^{AT}. Solving for H_T in (A.22) gives

$$H_T = A^{-2}\left(e^{AT} - I - AT\right) \tag{A.23}$$

which is Method 1 (3.10) as desired. Note that A is always invertible since it is an asymptotically stable matrix and by necessity has no zero eigenvalues.

Consider the well-known Laplace transform expression for the matrix exponential, i.e.,

$$\mathcal{L}\{e^{AT}\} = (sI - AT)^{-1} \tag{A.24}$$

Method 2 (3.11) follows by multiplying the Laplace transform in (A.24) by $1/s^2$ to correspond to a double integration in time, i.e.,

$$\mathcal{L}\{\int_0^T \int_0^{s'} e^{A\tau} d\tau ds'\} = \frac{1}{s^2}(sI - AT)^{-1} \tag{A.25}$$

■